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An extension of Christoffel duality to a subset of Sturm numbers and their characteristic words

Manuel Domínguez [a,*], Thomas Noll [b], Marco Castrillón [a]

[a] Universidad Complutense de Madrid, Departamento de Geometría y Topología, Spain

[b] ESMUC (Escola Superior de Musica de Catalunya, Barcelona), Departament de Teoria i Composició, Spain

A R T I C L E   I N F O

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A B S T R A C T

The paper investigates an extension of Christoffel duality to a certain family of Sturmian words. Given an Christoffel prefix w of length N of an Sturmian word of slope g we associate a *N-companion slope* g_N^* such that the upper Sturmian word of slope g_N^* has a prefix w^* of length N which is the upper Christoffel dual of w. Although this condition is satisfied by infinitely many slopes, we show that the companion slope g_N^* is an interesting and somewhat natural choice and we provide geometrical and music-theoretical motivations for its definition.

In general, the second-order companion $(g_N^*)_N^* = g_N^{**}$ does not coincide with the original g. We show that, given a rational number $0 < \frac{M}{N} < 1$, the map $g \to g_N^{**}$ has exactly one fixed point, $\phi_{\frac{M}{N}} \in [0, 1)$, called *odd mirror number*. We show that odd mirror numbers are Sturm numbers and their continued fraction expansion is purely periodic with palindromic periods of even length. The semi-periods are of odd length and form a binary tree in bijection to the Farey tree of ratios $0 < \frac{M}{N} < 1$. Its root is the singleton $\{2\}$, which represents the odd mirror number $\frac{-1+\sqrt{8}}{2} = [0; \overline{22}]$. The characteristic word $c_{\phi_{\frac{M}{N}}}$ of slope $\phi_{\frac{M}{N}}$ remains fixed under a standard morphism which can be computed from the semi-period of $\phi_{\frac{M}{N}}$. Finally, we prove that the characteristic word $G(c_{\phi_{\frac{M}{N}}})$ is a harmonic word.

As a minor open question we ask for the properties of *even mirror numbers*. A final conjecture provides a proper word-theoretic meaning to the extended duality for odd mirror number slopes: given a characteristic word $c_{\phi_{\frac{M}{N}}}$, the succession of those letters which immediately precede the occurrences of the left special factor of length N coincides – up to letter exchange – with the G-image of the dual word $c_{\phi^*_{\frac{M}{N}}}$.

1. Preliminaries

This paper elaborates the previous work on the junction of algebraic combinatorics on words on the one hand and algebraic musical scale theory on the other. Some connections were established in [12], where scale-step patterns of the so-called *well-formed musical scales* where shown to be Christoffel words. The theory of well-formed scales was developed by David Clampitt and Norman Carey in the late 80s and 90s [7,6,8] as a generalization of the main arithmetical and geometrical properties of the diatonic scale. These authors also introduced a notion of duality between scale-step patterns, which was demonstrated [12] to be analogous to the duality for Christoffel words (see [5]).

* Corresponding author. Tel.: +34 917187500.
E-mail address: mdrmanuel@gmail.com (M. Domínguez).

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Algebraic musical scale theory benefits from the knowledge and knowhow in algebraic combinatorics on words. Can this benefit also be reciprocal? Music-theoretical interest in musical modes already led to a refinement of Christoffel duality in [13] and [17]. The motivation for the present paper also emerged from the previous music-theoretical considerations (see Section 2.2). Yet the results do not offer an immediate music-theoretical interpretation. Therefore we count on the pure mathematical interest in the following question: given a Christoffel prefix w of length N of a Sturmian word s_g of slope g we ask whether there is a "natural" continuation of the dual Christoffel word w^* into a Sturmian word of a N-*dual* slope $g^* = g_N^*$? Proposition 5 provides an affirmative answer to this question. A concrete proposal for such a continuation of w^* is being made in Section 2. It turns out though, that the corresponding map $(w, s_g) \mapsto (w^*, s_{g_N^*})$ is not involutive and hence does not provide a proper duality in general. Section 3 is dedicated to the outstanding case, where the second-order companion slope g_N^{**} coincides with the original slope g. This case is interesting both from a number-theoretic and a word-theoretic point of view. In Section 1 we summarize basic facts about rational numbers, Möbius transforms, Christoffel words and their duals.

1.1. Farey numbers and $SL_2(\mathbb{N})$

Let $g = [0; a_1, a_2, \ldots]$ be the continued fraction expansion of a real number in the interval $I = [0, 1)$. Every rational number $\frac{M}{N} = [0; a_1, a_2, \ldots, a_{k-1}, b+1]$ with $0 < b \leq a_k$ is called *semi-convergent* of g.

Let us denote by $\mathfrak{F}$ the set of all fractions $\frac{M}{N}$ such that $0 \leq M < N$ and $gcd(M, N) = 1$. $\mathfrak{F}$ is called the set (or tree) of *Farey numbers*. It forms the left half the binary *Stern–Brocot tree* whose nodes comprise all positive rational numbers $r \in \mathbb{Q}^+$.

Every Farey number $\frac{M}{N}$ is determined by a path down the tree starting with a move to the left, i.e. by a word of the semi-group $L \cdot \{L, R\}^*$ (set of words in two letters, L and R, starting with letter L). The set of N-*Farey numbers* is the set $\mathfrak{F}_N = \{\frac{p}{q} \in \mathfrak{F}$ such that $q \leq N\}$. We are interested in triples of consecutive N-Farey numbers, that is, triples $\frac{M_1}{N_1} < \frac{M_2}{N_2} < \frac{M_3}{N_3}$ of fractions within $\mathfrak{F}_N$, where for every $\frac{p}{q} \in \mathfrak{F}_N \setminus \{\frac{M_1}{N_1}, \frac{M_2}{N_2}, \frac{M_3}{N_3}\}$ we have either $\frac{p}{q} < \frac{M_1}{N_1}$ or $\frac{M_3}{N_3} < \frac{p}{q}$.

The monoid $SL_2(\mathbb{N})$ of 2×2-matrices $L = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with natural number entries $a, b, c, d \in \mathbb{N}$ and determinant $ad - bc = 1$ is freely generated by the matrices $L = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$ and $R = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. The following number-theoretic facts about the conversion between nodes of the Stern–Brocot tree and paths from the root leading to these nodes are well known [14]:

Lemma 1. *The following three facts are equivalent:*

1. $\frac{M}{N} = [0; a_1, \ldots, a_k + 1]$.
2. $\frac{M}{N}$ *is the node associated with the path* $L^{a_1} R^{a_2} \cdots A^{a_k}$ *in the Stern–Brocot tree, where* $A = L$ *if* k *is odd and* $A = R$ *otherwise.*
3. $L^{a_1} R^{a_2} \cdots A^{a_k} = \begin{pmatrix} M_2 & M_1 \\ N_2 & N_1 \end{pmatrix}$ *with* $\frac{M}{N} = \frac{M_1+M_2}{N_1+N_2}$ *and* $\frac{M_1}{N_1} < \frac{M}{N} < \frac{M_2}{N_2}$ *are three consecutive* N*-Farey numbers.*

Example. (This example is continued throughout the paper in several stages.) Let us consider the 7-Farey number $\frac{4}{7}$, the continued fraction expansion of which is $[0; 1, 1, 3] = [0; 1, 1, 2+1]$. The path on the Stern–Brocot tree leading to $\frac{4}{7}$ is LRL^2. The corresponding $SL_2(\mathbb{N})$-matrix is $\alpha_{\frac{4}{7}} = LRL^2 = \left(\begin{smallmatrix} 3 & 1 \\ 5 & 2 \end{smallmatrix}\right)$. $\frac{1}{2} < \frac{4}{7} < \frac{3}{5}$ are three consecutive 7-Farey numbers.

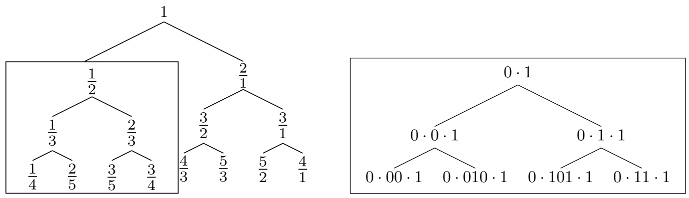
1.2. The Möbius transform

With every matrix $\beta \in GL_2(\mathbb{N}) \subset GL_2(\mathbb{C})$ we associate the linear fractional (or Möbius) transform of the extended complex plane $\mathbb{C} \cup \{\infty\} \overset{\textbf{def}}{=} \widehat{\mathbb{C}}$

$$\mu[\beta] : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}} \quad \text{with } \mu\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right](z) = \frac{az+b}{cz+d}.$$

Let $J = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ denote the matrix whose associated linear fractional transform is the reciprocal map: $\mu[J](z) = \frac{1}{z}$. The following facts are easily checked:

1. $\mu : GL_2(\mathbb{Z}) \to Aut\left(\widehat{\mathbb{C}}\right)$ is a group homomorphism, and its restriction $\mu : GL_2(\mathbb{N}) \to Aut\left(\widehat{\mathbb{C}}\right)$ is a monoid homomorphism.
2. $\mu[L](g) = \frac{g}{1+g} = \frac{1}{1+\frac{1}{g}} \Rightarrow \begin{cases} \mu[L]([0; a_1, a_2, \ldots]) = [0; a_1+1, a_2, \ldots] \\ \mu[L]([a_0; a_1, \ldots]) = [0; 1, a_0, a_1, \ldots] \quad \forall a_0 > 0. \end{cases}$
3. From 1 and 2, $\mu[L^k]([a_0; a_1, a_2, \ldots]) = \begin{cases} [0; a_1+k, a_2, \ldots] & \text{if } a_0 = 0 \\ [0; k, a_0, a_1, a_2, \ldots] & \text{if } a_0 \neq 0. \end{cases}$
4. $\mu[R](g) = 1 + g \Rightarrow \mu[R^k]([a_0; a_1, a_2, \ldots]) = [a_0 + k; a_1, a_2, \ldots] \quad a_o \geq 0.$
5. $\mu[J](g) = \frac{1}{g} \Rightarrow \begin{cases} \mu[J]([0; a_1, a_2, \ldots]) = [a_1; a_2, \ldots] \\ \mu[J]([a_0; a_1, \ldots]) = [0; a_0, a_1, \ldots] \quad \forall a_0 > 0. \end{cases}$



(a) Stern–Brocot tree. (b) Positive Christoffel words.

Fig. 1. Connection between positive Christoffel words and Farey numbers.

1.3. Mechanical words, Christoffel words and duality.

The *lower* (respectively *upper*) *mechanical word* of slope g and zero intercept is the infinite word $s_g : \mathbb{N} \to \{0, 1\}$ (respectively $\bar{s}_g : \mathbb{N} \to \{0, 1\}$) defined by

$$s_g(n) = \lfloor g(n+1) \rfloor - \lfloor gn \rfloor \quad (\text{respectively} \quad \bar{s}_g(n) = \lceil g(n+1) \rceil - \lceil gn \rceil).$$

Whenever the slope g is irrational, the lower and upper mechanical words s_g and $\bar{s}_g$ differ only in their very first letter. One has $s_g = 0c_g$ and $\bar{s}_g = 1c_g$, where c_g denotes the infinite *characteristic word* of slope g ($c_g : \mathbb{N} \backslash \{0\} \to \{0, 1\}$ with $c_g(n) = \lfloor g(n+1) \rfloor - \lfloor gn \rfloor$). If the slope $g = \frac{p}{n}$ is rational (with numerator n and denominator $p < n$ being co-prime), s_g (respectively $\bar{s}_g$) is an infinite periodic word and its primitive period w of length n is called a *lower Christoffel word* (respectively an *upper Christoffel word*) of slope $\frac{p}{n}$. One has that $p = |w|_y$ and $n = |w|$. Every lower Christoffel word w decomposes as $w = 0v1$ with v being a *central palindrome* (also called *central word*). Every upper Christoffel word can be written as $w = 1v0$, where $w^t = 0v1$ is the corresponding lower Christoffel word of the same slope $\frac{p}{n}$. Christoffel words are one of the first notions to appear in Combinatorics of Words. See [3] for a historical overview, or [2] for an extensive study of manifold characterizations and applications of Christoffel words.

Now we recall a procedure to generate central words. Let $\tilde{u}$ denote the mirror image of u, where letters are written in reversed order. Let $w \in \{0, 1\}^*$ be a word and let us write $w = uv$, where v is the longest suffix of w that is a palindrome. Then the *right palindromic closure* $w^+ = w\tilde{u}$ is the unique shortest palindrome having w as a prefix (see [11], Lemma 5). The *right iterated palindromic closure* of w is denoted by $PAL(x)$ and is defined recursively by $PAL(\epsilon) = \epsilon$ and $PAL(w) = (PAL(u)z)^+$, where $w = uz$ and z is the last letter of w. The image of the right iterated palindromic closure $PAL : \{0, 1\}^* \to \{0, 1\}^*$ coincides with the set of central words ([11], Proposition 8). If $PAL(v) = w$, then v is called the directive word of w. Fig. 1 displays in a tree diagram the set of Christoffel words $0 \cdot PAL(v) \cdot 1$, where $v \in \{0, 1\}^*$.

There is another characterization of central words by means of the set *PER* of all finite words w on the alphabet $\{0, 1\}^*$ having two periods p, q which are co-prime and such that $|w| = p + q - 2$. Recall that a positive integer p is a period of a finite word $w = a_1a_2 \ldots a_r$ if $a_i = a_{i+p}$ for all $1 < i < r - p$. From [11], Proposition 7, the set of central words coincide with the set *PER*.

We can fit the Farey tree (left half of Stern–Brocot tree) into the Christoffel tree if we shift it one step upwards. Following this observation and Lemma 1 we obtain:

Lemma 2. *The slope of* $0PAL(0^{h_1-1}1^{h_2}0^{h_3} \ldots i^{h_k})1$ *is* $[0; h_1, h_2, \ldots, h_k + 1]$, *where* $i = 0$ *if* k *is odd and* 1 *otherwise.*

Given a lower (respectively upper) Christoffel word w of slope $\frac{M}{N}$, we call the lower (respectively upper) Christoffel word of slope $\frac{M^*}{N}$ (where $M^* = M^{-1} \mod N$) the corresponding *dual Christoffel word*.

Let $\frac{M}{N} = [0; a_1, a_2, \ldots, a_k + 1] \in \mathfrak{F}$. In accordance with Lemma 2, the corresponding lower Christoffel word decomposes as $0PAL(0^{a_1-1}1^{a_2} \cdots i^{a_k})1$. Recall from [5], Proposition 3.1, that directive words of mutually dual central words are mutually reversed. Thus, $0PAL(i^{a_k} \cdots 1^{a_2}0^{a_1-1})1$ is the corresponding Christoffel dual word. The proof of Lemma 3 below is straight forward:

Lemma 3. *Let* $\frac{M}{N} \in \mathfrak{F}$. *Then:*

$$\frac{M}{N} = [0, a_1, a_2, \ldots, a_{k-1}, a_k + 1] \Rightarrow \frac{M^*}{N} = \begin{cases} [0;\, 1, a_k, a_{k-1}, \ldots, a_1] & \text{if } k \text{ is even} \\ [0;\, 1 + a_k, a_{k-1}, \ldots, a_1] & \text{if } k \text{ is odd}. \end{cases}$$

Example (*Continuation*)**.** The lower Christoffel word of slope $\frac{4}{7}$ is $0PAL(0^01^10^2)1 = 0PAL(100)1 = 0101011$. Then $0PAL(001)1 = 0001001$ is the corresponding dual Christoffel word, that is, the lower Christoffel word of slope $\frac{2}{7}$. The matrix associated with $\frac{2}{7} = [0; 1 + 2, 1, 1] = [0; 3, 1, 1]$ is $\alpha_{\frac{2}{7}} = L\widetilde{RL^2} = L^3R = \left(\begin{smallmatrix}1 & 1\\ 3 & 4\end{smallmatrix}\right)$.

2. The *N*-companion slope

In preparation of the technical definition in Section 2.3 we briefly recapitulate some math–music-theoretical background (Section 2.1) and the concrete motivation (Section 2.2) which eventually led to the *N*-Companion Slope.

Fig. 2. The lower Pythagorean word s_g of slope $g = \log_2(\frac{3}{2})$ encodes the folding of an infinite chain of ascending fifths into a pitch height ambit of on octave (see top system). The letter 0 stands for ascending fifths and the letter 1 stands for descending fourths. In the scale-step pattern of diatonic scale (bottom left) 0 stands for ascending major seconds and 1 for ascending minor seconds. In the scale-step pattern of chromatic scale (bottom right) 0 stands for ascending augmented primes and 1 for ascending minor seconds.

2.1. Music-theoretical background: Well-formed scales

The *Pythagorean word* $s_g = 0101011010110101011010110101011010110101011\ldots$ with $g = \log_2(\frac{3}{2})$ is a prominent mechanical word in music theory (see Fig. 2). The prefixes of lengths $N = 7$ and $N = 12$ are lower Christoffel words and correspond to the folding patterns of the diatonic and chromatic tone repertoires, respectively. Their Christoffel duals encode the associated ascending scale-step patterns (see bottom systems). The concrete slope $\log_2(\frac{3}{2})$ represents the relative pitch height of the musical interval of a *fifth* compared to the pitch height of the musical interval of an *octave*, which can be normalized to the value $1 = \log_2(\frac{2}{1})$. Christoffel prefixes of this infinite word correspond to *well-formed musical scales*, i.e. to tone repertoires whose orderings in ascending fifths can be permuted into ascending pitch height order by virtue of a linear automorphism of $\mathbb{Z}_N$. Christoffel duality provides a theoretically rich and precise account to the constitution of tone relations and may eventually replace the traditional distinction between tone kinship and pitch height neighborhood.

Consider two sequences with entries in the interval $[0, 1) \subset \mathbb{R}$, namely $\sigma = (\sigma_0, \ldots, \sigma_{N-1})$ and $\Gamma(N, g) = (\gamma_0, \ldots, \gamma_{N-1})$, where $\gamma_k = kg \bmod 1$ for $k = 0, \ldots, N-1$. $\Gamma(N, g)$ yields an arithmetic sequence under projection to the circle group $\mathbb{R}/\mathbb{Z}$ starting from 0. Assume that the entries of σ appear in ascending order $0 = \sigma_0 < \sigma_1 < \cdots < \sigma_{N-1} < 1$. Further suppose that the entries from both sequences form the same set:

$$\{\sigma_0, \ldots, \sigma_{N-1}\} = \Gamma(N, g).$$

Hence, there is a permutation $\pi : \{0, \ldots, N-1\} \to \{0, \ldots, N-1\}$ such that $\gamma_k = \sigma_{\pi(k)}$ for $k = 0, \ldots, N-1$. Let $M = \pi(1)$ denote the index of the entry γ_1 within σ. The following proposition paraphrases the basic mathematical idea in Norman Carey and David Clampitt's paper [7]:

Proposition 1. *With the above notation the following conditions are equivalent:*

1. *The ratio $\frac{M}{N}$ is a semi-convergent of g.*
2. *(symmetry condition) $\pi(k) = Mk \bmod N$ for $k = 0, \ldots, N-1$.*
3. *(closure condition) $0 \le Ng \bmod 1 < \sigma_1$ or $\sigma_{N-1} < Ng \bmod 1 < 1$.*

Regardless of the conditions in Proposition 1, the points $s_0, \ldots, s_{N-1}$ partition the unit circle into N intervals having at most three lengths (see the *Three distance theorem* in [4] or [1]). Under the stronger conditions of Proposition 1 there are only two distances $\{M^* \cdot g\}$ and $\{1 - (N - M^*) \cdot g\}$ (with $M^* = M^{-1} \bmod N$) and the partition yields a Christoffel word w of slope $\frac{M^*}{N}$ in letters 0 and 1 (see [12]). The definition is sensitive with respect to the side of the semi-convergent: for $\frac{M}{N} \le g$ one puts

$$w_k = \begin{cases} 0, & \text{if } \sigma_{k+1} - \sigma_k = \{M^* \cdot g\} \\ 1, & \text{if } \sigma_{k+1} - \sigma_k = \{1 - (N - M^*) \cdot g\} \end{cases}$$

and obtains a lower Christoffel word. For $g < \frac{M}{N}$ it is preferable to read the above word backwards and to obtain an upper Christoffel word. Music-theoretically, the equivalent conditions in Proposition 1 characterize *well-formed scales* and these words w encode ascending (or descending) *scale-step patterns*. In the prominent musical application $g = \log_2(\frac{3}{2})$ and Fig. 2 illustrates, how this relates to the lower *Pythagorean word* s_g and its lower Christoffel prefixes of length $N = 7$ and $N = 12$. For example, the three consecutive 7-Farey numbers in association with $\frac{M}{N} = \frac{4}{7}$ are $\frac{1}{2} < \frac{4}{7} < \frac{3}{5}$ and as long as the slope g satisfies $\frac{4}{7} \le g < \frac{3}{5}$, the situation remains qualitatively the same. However, the famous pentatonic scale is associated with the larger semi-convergent $\frac{3}{5}$. Here one needs to consider the upper Pythagorean word $\bar{s}_g = 11010110101101010110\ldots$ instead, with the upper Christoffel prefix 11010 of length 5 and slope $\frac{3}{5}$. The dual upper Christoffel word 10100 of slope $\frac{2}{5}$ encodes the scale-step pattern of the descending pentatonic scale $F - D - C - A - G - (\underline{F})$. Proposition 2 keeps track of this circumstance.

Proposition 2. *Consider a slope $g \in (0, 1)$ with semi-convergent $\frac{M}{N}$ and fix three consecutive N-Farey numbers $\frac{M_1}{N_1} < \frac{M}{N} < \frac{M_2}{N_2}$.*

1. *If $g \in [\frac{M}{N}, \frac{M_2}{N_2})$, the ascending scale-step pattern w of $\Gamma(N, g)$ and the prefix of the lower mechanical word s_g of length N are mutually dual lower Christoffel words.*

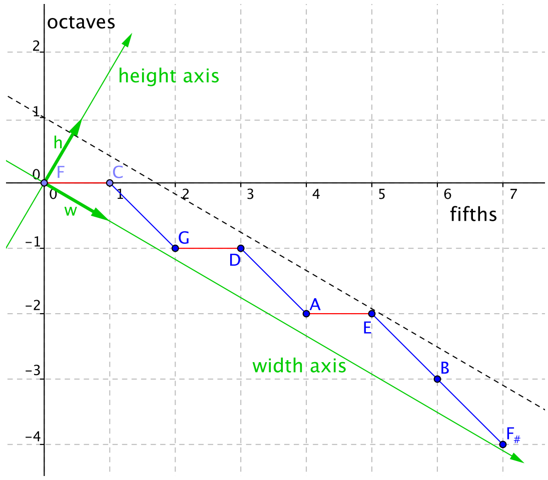


Fig. 3. The fifth-up/fourth-down trajectory fits to the width axis from above and yields the lower Pythagorean word.

2. *If* $g \in (\frac{M_1}{N_1}, \frac{M}{N})$, *the descending scale-step pattern* w *of* $\Gamma(N, g)$ *and the prefix of the upper mechanical word* $\bar{s}_g$ *of length* N *are mutually dual upper Christoffel words.*

2.2. Music-theoretical motivation: Height–width duality

Despite the nice duality for every single semi-convergent of the Pythagorean word, there is an asymmetry in Fig. 2. The folding pattern into ascending fifths and descending fourths (top system) is an infinite word of irrational slope $g = \log_2(\frac{3}{2})$, while the scale-step patterns (bottom systems) are just finite words, which – at the utmost – yield octave-periodic extensions with rational slopes $\frac{2}{7}$ or $\frac{7}{12}$. Is it possible to get rid of this asymmetry?

Consider the musical intervals of *fifth* x and *octave* o as generators of a free commutative group $\mathcal{P} = \mathbb{Z}x \oplus \mathbb{Z}o$ of range 2. Any linear combination $ax+by$ yields a composed musical interval. The attribution of an interval size in terms of pitch height is suitably described in terms of a linear form $\mathfrak{h} : \mathcal{P} \to \mathbb{R}$ with $\mathfrak{h}(a, b) := a+gb$. The basic pitch heights of fifth and octave are $\mathfrak{h}(x) = g = \log_2(\frac{3}{2})$ and $\mathfrak{h}(o) = 1$. The linear form $\mathfrak{h}$ can be linearly extended to the real vector space $\mathbb{R}x \oplus \mathbb{R}o$ where we can inspect the vector $w = (-g, 1)$, generating the kernel $ker(\mathfrak{h})$ of $\mathfrak{h}$ as well as its gradient $v = grad(\mathfrak{h}) = (1, g)$. Sometimes music scholars are content with simply working in the image $\mathfrak{h}(\mathcal{P}) \subset \mathbb{R}$, arguing that $\mathcal{P}$ and $\mathfrak{h}(\mathcal{P})$ are isomorphic as groups anyway. They oversee the fact that $\mathcal{P}$ is a discrete subgroup of $\mathbb{R}^2$, while $\mathfrak{h}(\mathcal{P})$ is dense everywhere in $\mathbb{R}$. If one identifies the pitch height dimension $\mathfrak{h}(\mathcal{P})$ with the subspace $\mathbb{R}v \subset \mathbb{R}^2$, one may boil down the difference between the alternative views to the question, whether or not the kernel $ker(\mathfrak{h}) = \mathbb{R}w$ proves to be music-theoretically relevant. The previous Section 2.1 can easily be interpreted as a pledge for an affirmative answer: Fig. 3 illustrates the role of the equivocal subspace $ker(\mathfrak{h})$, which is called the *pitch width dimension* in addition to the usually unquestioned pitch height dimension. The mechanical Pythagorean word s_g encodes the unique trajectory of those $\mathcal{P}$-points being closest approximations to the width axis from above.

Likewise one may geometrically interpret the periodic scale-step patterns as encodings of mechanical trajectories along the octave axis with respect to a lattice, which is generated by the two step intervals. This observation is the starting point for our attempt to remove the asymmetry between the infinite Pythagorean word s_g with an irrational slope and the periodic scale-step patterns. While keeping the step interval lattice as a frame for mechanical trajectories one may replace the octave axis in its role as a fitting line for scale-step trajectory by the height axis. This leads to a well-defined irrational slope and its definition is given in the following Section 2.3 (see also [10]).

2.3. Formal definition of the N-companion slope g_N^*

Definition. The **companion function** $\Psi : I \times \mathfrak{F} \longrightarrow I$ associates with every slope $h \in I = [0, 1)$ and every Farey number $\frac{M}{N}$ the slope $\Psi\left(h, \frac{M}{N}\right) := \frac{M_1 h + N_1}{Mh+N}$. Here $\frac{M_1}{N_1} < \frac{M}{N}$ are two consecutive N-Farey numbers.
When $\frac{M}{N}$ is a semi-convergent of g, we write $g_N^* := \Psi\left(g, \frac{M}{N}\right)$ and call g_N^* the N-**companion** of g.

If $\alpha = \left(\begin{smallmatrix} M_2 & M_1 \\ N_2 & N_1 \end{smallmatrix}\right)$ is the matrix of $SL_2(\mathbb{N})$ associated with $\frac{M}{N}$ (see Lemma 1), then it is immediate to check that

$$(\alpha \cdot L \cdot J)^t = \begin{pmatrix} M_1 & N_1 \\ M & N \end{pmatrix} \Rightarrow \Psi\left(h, \frac{M}{N}\right) = \mu[(\alpha \cdot L \cdot J)^t](h).$$

Now we relate the value of $\Psi\left(h, \frac{M}{N}\right)$ with the continued fraction expansion of h and $\frac{M}{N}$. For that purpose we need the following lemma, which is an immediate consequence of the properties of the Möbius transform seen in Section 1.2:

Lemma 4. *If $P = L^{b_1}R^{b_2}\cdots R^{b_{2k}}L^{b_{2k+1}}J$ with $b_{2k+1} \geq 0$, $b_i > 0 \quad \forall i = 1, \ldots 2k$, and $g = [0; a_1, a_2, \ldots]$ then one finds*

$$\mu[P](g) = [0; b_1, b_2, \ldots, b_{2k+1} + a_1, a_2, \ldots].$$

Proposition 3. *With $\frac{M}{N} = [0; a_1, \ldots a_k + 1]$ and $h = [0; b_1, b_2, \ldots]$ one obtains:*

$$\Psi\left(h, \frac{M}{N}\right) = \begin{cases} [0; a_k + 1, a_{k-1}, \ldots, a_1, b_1, b_2, \ldots] & \text{if } k \text{ is odd} \\ [0; 1, a_k, a_{k-1}, \ldots, a_1, b_1, b_2, \ldots] & \text{if } k \text{ is even.} \end{cases}$$

Proof. Let α be the matrix of $SL_2(\mathbb{N})$ associated with $\frac{M}{N}$. Then $\Psi\left(h, \frac{M}{N}\right) = \mu[(\alpha \cdot L \cdot J)^t](h)$. If k is odd,

$$(\alpha \cdot L \cdot J)^t = \left(L^{a_1}R^{a_2}\cdots L^{a_k+1}J\right)^t = JR^{a_k+1}\cdots L^{a_2}R^{a_1} = L^{a_k+1}\cdots R^{a_2}L^{a_1}J.$$

If k is even,

$$(\alpha \cdot L \cdot J)^t = \left(L^{a_1}R^{a_2}\cdots R^{a_k}LJ\right)^t = JRL^{a_k}\cdots L^{a_2}R^{a_1} = LR^{a_k}\cdots R^{a_2}L^{a_1}J,$$

and we close the proof on the basis of Lemma 4. □

2.4. A geometric description of g_N^*

Let us consider the line r satisfying the equation $r \equiv y = gx$ and let

$$S(N, g) = \{P_k = (k, \lfloor k \cdot g \rfloor)\}_{k=0,1,\ldots,N-1}$$

be the first N points of the corresponding lower mechanical sequence. In this subsection we will see how the well-formed sets $\Gamma(g, N)$ and $\Gamma(g_N^*, N)$ defined just can be generated from the prefix of length N of the mechanical word s_g by means of a pair of projections with direction parallel and perpendicular to r. For that purpose, the well-formed sets $\Gamma(g, N)$ will be displayed as set of points within certain intervals $[a, b)$, rather than the unit circle, as they were defined just before Proposition 1.

Let s be a second line through the origin $O = (0, 0)$, different from $y = gx$ and from the perpendicular $y = -\frac{1}{g}x$, and let finally Π_s and Π_s' be the parallel and perpendicular projections with respect to the line $y = gx$ onto the line s:

$$\begin{array}{rccc} \Pi_s: & \mathbb{R}^2 & \longrightarrow & s \\ & P & \longmapsto & r_P \cap s \end{array} \qquad \begin{array}{rccc} \Pi_s': & \mathbb{R}^2 & \longrightarrow & s \\ & P & \longmapsto & r_P' \cap s \end{array},$$

where r_P (resp. r_P') denotes the line parallel to r (resp. perpendicular to r) through a given point P.

Now, let us study the behavior of the set of points $S(N, g)$ when we project it onto s by means of Π_s and Π_s'. For that purpose N will be the denominator of a semi-convergent $\frac{M}{N}$ of g, such that $\Gamma(g, N)$ is a well-formed set and the prefix of length N of s_g is a lower Christoffel word.

Proposition 4. *With the previous notation the following assertions are true:*

1. *$\Pi_s S(g, N)$ and the well-formed set $\Gamma(g, N)$ are congruent.*
2. *$\Pi_s' S(g, N)$ and the well-formed set $\Gamma(g_N^*, N)$ are congruent.*

Proof. 1. The mechanical sequence of slope g is in the region of the plane delimited by the lines $y = gx$ and $y = gx - 1$. Therefore the projection of every point of that sequence is in the segment of r determined by $\Pi_s(O)$ and $\Pi_s(Q)$. Moreover, one has by Thales that

$$\frac{\overline{O\Pi_s(P_1)}}{\overline{O\Pi_s(Q)}} = g \quad \text{see Fig. 4.}$$

Notice that $\Pi_s S(g, N)$ is congruent with the projection of $S(g, N)$ onto the ordinate axis. This last set coincides with the set

$$-\Gamma(g, N) = \{-\{k * g\}, k = 0, 1, \ldots, N\} \quad \text{see Fig. 4.}$$

It follows that the set $\Pi_s S(g, N)$ and the set $\Gamma(N, g)$ are congruent.

2. Now, let us consider the projection $\Pi_s' S(g, N)$ onto the segment $\overline{O\Pi_s'(N, M)}$ of r. Notice how Π_s' preserves the pattern of $S(g, N)$, so by Proposition 2 one has that the step pattern of $\Pi_s' S(g, N)$ coincides with the Christoffel word of slope $\frac{M^*}{N}$.

If we project by means of Π_s the first step of $S(g, N)$, that is $\overline{OP_1}$, onto s we get the *generator* $\overline{O\Pi_s P_1} = g$. Therefore, if we want to compute the new generator, we need to project by means of Π_s' the new *first step*, which coincides with

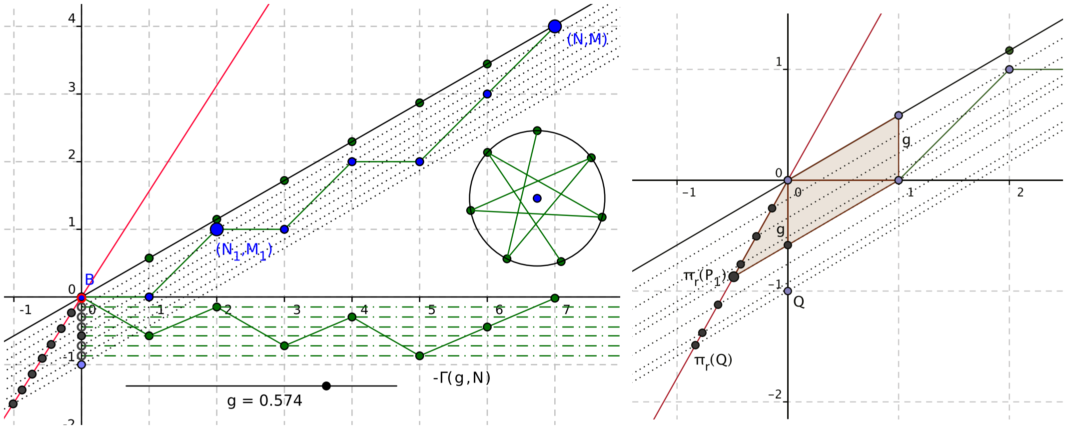


Fig. 4. Parallel projection of a mechanical sequence.

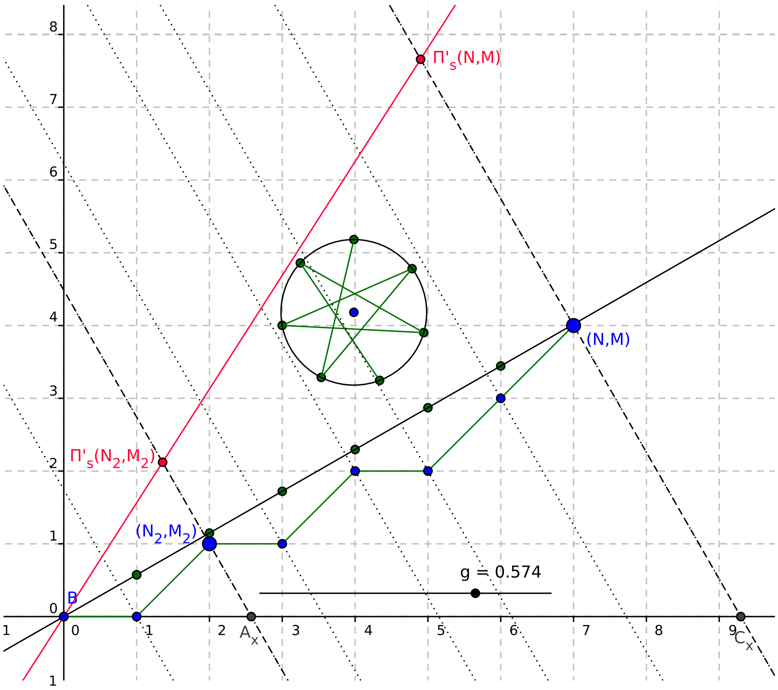


Fig. 5. Perpendicular projection of a mechanical sequence.

the segment that joins the origin O with the point closest to the line $y = gx$. From [2] Lemma 1.3, we know that this point is the one which divides the Christoffel word of slope $\frac{M}{N}$ into a Christoffel pair (w_1, w_2) and its coordinates are $(|w_1|_0, |w_1|_1) = (N_1, M_1) = P_{N_1}$.

Let r_{N_1} and r_N be the lines of slope $\frac{-1}{g}$ passing through P_{N_1} and P_N respectively. Their equations are given by:

$$r_{N_1} \equiv y = \frac{-1}{g}x + M_1 + \frac{N_1}{g} \qquad r_N \equiv y = \frac{-1}{g}x + M + \frac{N}{g}.$$

We may compute *the new generator* with the projections of P_{N_2} and (N, M) over the horizontal axis, by Thales (see Fig. 5):

$$g^* = \frac{\overline{P_0 \Pi_s'(P_{N_1})}}{\overline{P_0 \Pi_s'(P_N)}} = \frac{\overline{OA}}{\overline{OC}} = \frac{M_1 + \frac{N_1}{g}}{M + \frac{N}{g}} = \frac{M_1 g + N_1}{Mg + N}. \quad \square$$

Proposition 5. *Let $\bar{s}_g$ be an upper mechanical sequence of slope $g \in [0, 1)$ and let $\frac{M}{N}$ be a semi-convergent of g. Moreover, let $1u0$ be the upper Christoffel prefix of length N and of slope $\frac{M}{N}$ of $\bar{s}_g$. Then the upper Sturmian word $\bar{s}_{g_N^*}$ of slope g_N^* has a prefix w^* of length N which is the upper Christoffel dual of w.*

Proof. Let $\alpha = \left(\begin{smallmatrix} M_2 & M_1 \\ N_2 & N_1 \end{smallmatrix}\right)$ and $\alpha^* = \left(\begin{smallmatrix} M_2^* & M_1^* \\ N_2^* & N_1^* \end{smallmatrix}\right)$ be, respectively, the matrices associated with $\frac{M}{N}$ and $\frac{M^*}{N}$. Then one can check that

$$\Psi\left(g, \frac{M}{N}\right) \in \left(\frac{M_1^*}{N_1^*}, \frac{M^*}{N}\right) \quad \forall g \in \left(\frac{M_1}{N_1}, \frac{M_2}{N_2}\right)$$

and we close the proof on the basis of Proposition 2. □

3. Duality: fixed points under the map $g \to g_N^{**}$

The N-companion of a fraction $\frac{M}{N}$ does not coincide with the Christoffel dual slope $\frac{M^*}{N} = \frac{M^{-1} \mod N}{N}$, that is, $\left(\frac{M}{N}\right)_N^* \stackrel{\textbf{def}}{=} \Psi\left(\frac{M}{N}, \frac{M}{N}\right) \neq \frac{M^*}{N}$. The following proposition asserts that $\frac{M^*}{N}$ is a "good" approximation of $\left(\frac{M}{N}\right)_N^*$ and it is a straightforward consequence of Proposition 3 and Lemma 3.

Proposition 6. *For every $\frac{M}{N} \in \mathfrak{F}$, $\frac{M^*}{N}$ is a semi-convergent of $\left(\frac{M}{N}\right)_N^*$.*

Example (*Continuation*)**.** The 7-companion of $\frac{4}{7}$ is $\left(\frac{4}{7}\right)_7^* = [0; 3, 1, 1, 1, 1, 3] = \frac{18}{65}$. This value can be computed directly from the definition of Ψ. Recall that, in this case, $\frac{M_1}{N_1} = \frac{1}{2}$:

$$\Psi\left(h, \frac{4}{7}\right) = \frac{h+2}{4h+7} \quad \text{and, thus} \quad \left(\frac{4}{7}\right)_7^* = \Psi\left(\frac{4}{7}, \frac{4}{7}\right) = \frac{\frac{4}{7}+2}{4\frac{4}{7}+7} = \frac{18}{65}.$$

Notice that $\frac{2}{7}$ is a semi-convergent of $\frac{18}{65}$.

As an immediate consequence of the previous proposition, the mechanical words of slopes $\frac{M}{N}$ and its N-companion $\left(\frac{M}{N}\right)_N^*$ have Christoffel dual prefixes of length N.

Notice that $\frac{M^*}{N}$ is a semi-convergent of g_N^* whenever $\frac{M}{N}$ is a semi-convergent of g. Therefore it is convenient to attach this information to the definition of Ψ and define a map: $\widetilde{\Psi} : I \times \mathfrak{F} \longrightarrow I \times \mathfrak{F}$ with $\widetilde{\Psi}(h, \frac{M}{N}) := (\Psi(h, \frac{M}{N}), \frac{M^*}{N})$. This map can be composed with itself and we obtain the second *second-order companion function*: $\widetilde{\Psi} \circ \widetilde{\Psi} : I \times \mathfrak{F} \longrightarrow I \times \mathfrak{F}$. Whenever $\frac{M}{N}$ is a semi-convergent for g we write $\widetilde{\Psi} \circ \widetilde{\Psi}(g, \frac{M}{N}) := (g_N^{**}, \frac{M}{N})$ and call g_N^{**} the *second-order N-companion* of g. The map $g \mapsto g_N^{**}$ is defined in the open interval $(\frac{M_1}{N_1}, \frac{M_2}{N_2})$ containing $\frac{M}{N}$ and is natural to look for its fixed points. To that end we need the formula of Ψ with respect to a dual Farey number $\frac{M^*}{N}$.

Lemma 5. *Let α be the matrix associated with the Farey number $\frac{M}{N}$. Then*

$$\Psi\left(g, \frac{M^*}{N}\right) = \mu\left[\alpha LJ\right](g).$$

Proof. Notice that dual Farey numbers have retrograde paths in the left half of the Stern–Brocot tree. Moreover, if $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is a matrix of $SL_2(\mathbb{N})$ and $\tilde{\gamma} = \left(\begin{smallmatrix} d & b \\ c & a \end{smallmatrix}\right)$ (main diagonal flipped), then γ and $\tilde{\gamma}$ can be written as retrogradation words in $\{L, R\}^*$ (see [14]). One can also check that $J\tilde{\gamma}^t J = \gamma$ and finally we have:

$$\Psi\left(g, \frac{M^*}{N}\right) = \mu\left[(L\tilde{\gamma}LJ)^t\right](g) = \mu\left[JR\tilde{\gamma}^t R\right](g) = \mu\left[L\gamma LJ\right](g) = \mu\left[\alpha LJ\right](g). \quad \square$$

As Proposition 7 will state, the only fixed numbers under $g \mapsto g_N^{**}$ are those given by the following definition:

Definition. With every Farey number $\frac{M}{N} = [0; a_1, \ldots, a_k + 1]$ we associate the number $\phi_{\frac{M}{N}}$ by virtue of

$$\phi_{\frac{M}{N}} := \begin{cases} [0; \overline{a_1, \ldots, a_k + 1, 1 + a_k, \ldots, a_1}] & \text{if } k \text{ is odd} \\ [0; \overline{a_1, \ldots, a_k, 1, 1, a_k, \ldots, a_1}] & \text{if } k \text{ is even.} \end{cases}$$

This number $\phi_{\frac{M}{N}}$ is called **odd mirror number**, since its continued fraction expansion is palindromic with a period of odd length. The **semi-period** of an odd mirror number $\phi_{\frac{M}{N}}$ (the first half of its period) is denoted by $a_{\frac{M}{N}}$. In other words, $a_{\frac{M}{N}} = (a_1, \ldots, a_k + 1)$ if k is odd and $a_{\frac{M}{N}} = (a_1, \ldots, a_k, 1)$ if k is even.

Proposition 7 (*Characterization of Odd Mirror Numbers*)**.** *Fix three consecutive N-Farey numbers $\frac{M_1}{N_1} < \frac{M}{N} < \frac{M_2}{N_2}$. Among all $g \in (\frac{M_1}{N_1}, \frac{M_2}{N_2})$, i.e. among all numbers to which $\frac{M}{N}$ is a semi-convergent, $\phi_{\frac{M}{N}}$ is the only number for which N-companionship is involutive. Equivalently, $\phi_{\frac{M}{N}}$ is the only solution of the equation $g = g_N^{**}$ within $(\frac{M_1}{N_1}, \frac{M_2}{N_2})$.*

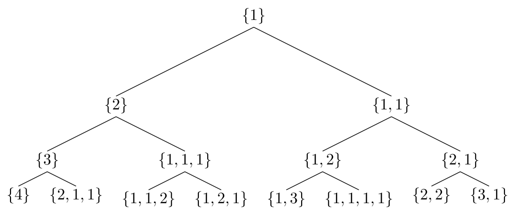


Fig. 6. Semi-period tree.

Table 1
Odd mirror numbers.

$\alpha_{\frac{M}{N}}$	$\phi_{\frac{M}{N}}$	*f. exp.*
$L \cdot Id$	$\frac{-1+\sqrt{8}}{2}$	$[0, \overline{22}]$
$L \cdot L$	$\frac{-3+\sqrt{13}}{2}$	$[0, \overline{33}]$
$L \cdot LL$	$\frac{-4+\sqrt{20}}{2}$	$[0, \overline{44}]$
$L \cdot R$	$\frac{-1+\sqrt{5}}{2}$	$[0, \overline{111111}]$
$L \cdot RR$	$\frac{-1+\sqrt{10}}{3}$	$[0, \overline{121121}]$
$L \cdot LR$	$\frac{-29+\sqrt{1517}}{26}$	$[0, \overline{211112}]$
$L \cdot RL$	$\frac{-19+\sqrt{1517}}{34}$	$[0, \overline{112211}]$

Proof. Let $\frac{M}{N} = [0; a_1, a_2, \dots, a_k + 1]$ and $g = [0; b_1, b_2, \dots]$. Let us suppose that k is even. Then one has that:

$$g_N^{**} = g \Leftrightarrow \mu[(\alpha LJ)(\alpha LJ)^t](g) = g.$$

We compute $\mu[(\alpha LJ)(\alpha LJ)^t](g)$:

$$\begin{aligned} \mu[(\alpha LJ)(\alpha LJ)^t](g) &= \mu[\alpha LJ]\mu[(\alpha LJ)^t](g) = \mu[\alpha LJ]([0; 1, a_k, \dots, a_1, b_1, b_2, \dots]) \\ &= [0; a_1, \dots, a_k, 1, 1, a_k, \dots, a_1, b_1, b_2, \dots]. \end{aligned}$$

The case of k odd is similar. $\square$

Remark 1 (*Recursive Construction of the Semi-Periods*)**.** *It is possible to translate the tree structure of $\mathfrak{F}$ into the set of semi-periods $a_{\frac{M}{N}} = \{a_1, \dots, a_{k-1}, a_k\}$ of odd mirror numbers. As a result of the definition, we have:*

- *The semi-period of the odd mirror number associated with the left son of the Farey number $M/N = [0; a_1, \dots, a_k + 1]$ is $\{a_1, \dots, a_{k-1}, a_k + 1\}$.*
- *The semi-period of the odd mirror number associated with the right son of the Farey number $M/N = [0; a_1, \dots, a_k + 1]$ is*
  - $\{a_1, \dots, a_{k-1}, a_k - 1, 1, 1\}$ *if* $a_k > 1$.
  - $\{a_1, \dots, a_{k-1} + 1, a_k\}$ *if* $a_k = 1$.

These recursive rules produce the left half of the tree displayed in Fig. 6. The main information about the corresponding odd mirror numbers is condensed in Table 1.

Remark 2. *For any $\frac{M}{N} \in \mathfrak{F}$ the map $\widetilde{\Psi} \circ \widetilde{\Psi}(\dots, \frac{M}{N}) : I \times \{\frac{M}{N}\} \to I \times \{\frac{M}{N}\}$ yields a contractive map (with respect to the euclidean distance) $\Psi^{**}_{\frac{M}{N}} : I \to I$. Therefore one may interpret $\phi_{\frac{M}{N}}$ as the limit*

$$\lim_{n\to\infty} (\Psi^{**}_{\frac{M}{N}})^n (g) = \phi_{\frac{M}{N}} \quad \textit{for every } g \in I.$$

Remark 3. *Recall that the incidence matrices of dual Christoffel words decompose as retrograde words in $\langle L, R\rangle$. Therefore an odd mirror number $\phi_{\frac{M}{N}}$ and its N-dual $\phi^*_{\frac{M}{N}}$ have retrograde semi-periods. In this sense, the extension of duality over the set of fixed slopes has a similar behavior as the duality of Christoffel words. If $\phi_{\frac{M}{N}} = \left[0; \overline{a_{\frac{M}{N}}, a^t_{\frac{M}{N}}}\right]$ is an odd mirror number, its N-dual has the continued fraction expansion: $\phi^*_{\frac{M}{N}} = \phi_{\frac{M^*}{N}} = \left[0; a^t_{\frac{M}{N}}, \overline{a_{\frac{M}{N}}, a^t_{\frac{M}{N}}}\right] = \left[0; \overline{a^t_{\frac{M}{N}}, a_{\frac{M}{N}}}\right]$.*

Remark 4. *The only number g that verifies* $g_N^{**} = g$ *for every N denominator of its semi-convergents is the decimal part of the golden number* $\phi = \left[0; \overline{1}\right]$, *which appears infinite many times in the semi-period tree (the nodes of the zig-zag paths* $L(RL)^k$*).*

Proposition 8. $\phi_{\frac{M}{N}}$ *is a Sturm number for every* $\frac{M}{N} \in \mathfrak{F}$. *If* $\left(\begin{smallmatrix} M_2 & M_1 \\ N_2 & N_1 \end{smallmatrix}\right)$ *is the matrix of* $SL_2(\mathbb{N})$ *associated with* $\frac{M}{N}$, *then it can be computed as:*

$$\phi_{\frac{M}{N}} = \frac{M_1^2 + M^2 - N_1^2 - N^2 + \sqrt{(M_1^2 + M^2 - N_1^2 - N^2)^2 + 4(N_1 M_1 + NM)^2}}{2(N_1 M_1 + MN)}.$$

Proof. Recall that $\phi_{\frac{M}{N}}$ is a solution of the equation $\mu[(\alpha LJ)(\alpha LJ)^t](g) = g$. The matrix $(\alpha LJ)(\alpha LJ)^t$ is symmetric. Thus, the previous equation is equivalent to:

$$\mu\left[\begin{pmatrix} A & B \\ B & C \end{pmatrix}\right](g) = g \Leftrightarrow \frac{A \cdot g + B}{B \cdot g + C} = g \Leftrightarrow Bg^2 + (C - A)g - B = 0,$$

where $A = M_1^2 + M^2$, $B = M_1 N_1 + NM$ and $C = N_1^2 + N^2$. Let $\phi_{\frac{M}{N}}$ and γ be the solutions of the previous equation. Then we have

$$\gamma \cdot \phi_{\frac{M}{N}} = -1 \Leftrightarrow \gamma = \frac{-1}{\phi_{\frac{M}{N}}} < 1.$$

One may conclude that $\phi_{\frac{M}{N}}$ is a Sturm number (see [15], Theorem 2.3.26). □

Example (*Continuation*)**.** The semi-period that corresponds with $\frac{4}{7} = [0; 1, 1, 3]$ is $a_{\frac{4}{7}} = \{1, 1, 3\}$, which leads to the odd mirror number $\phi_{\frac{4}{7}} = \left[0; \overline{1, 1, 3, 3, 1, 1}\right] = \frac{-3+\sqrt{34}}{5}$. The corresponding 7-dual is $\phi^*_{\frac{4}{7}} = \phi_{\frac{2}{7}} = \left[0; \overline{3, 1, 1, 1, 1, 3}\right] = \frac{-5+\sqrt{34}}{3}$.

Let $G(0, 1) = (0, 01)$, $D(0, 1) = (10, 1)$ and $E(0, 1) = (1, 0)$ denote the classical generators of the standard Sturmian monoid. Following [15] Theorem 2.3.25, the characteristic word of slope $\phi_{\frac{M}{N}}$ is fixed by some no trivial standard Sturmian morphism. Next proposition computes it in terms of G, D and E:

Proposition 9. *Let* $\phi_{\frac{M}{N}}$ *be an odd mirror number with semi-period* $a_{\frac{M}{N}} = \{a_1, \ldots, a_k\}$. *The standard morphism* $f_{\frac{M}{N}}$ *that fixes the characteristic word of slope* $\phi_{\frac{M}{N}}$ *decomposes in the following way:*

$$f_{\frac{M}{N}} = i_{\frac{M}{N}} \cdot j_{\frac{M}{N}} \quad \textit{with} \begin{cases} i_{\frac{M}{N}} = G^{a_1 - 1} \cdot D^{a_2} \cdots G^{a_k} \cdot D \cdot E \\ j_{\frac{M}{N}} = G^{a_k - 1} \cdot D^{a_{k-1}} \cdots G^{a_1} \cdot D \cdot E. \end{cases}$$

Proof. Notice that $f_{\frac{M}{N}}$ is locally characteristic, and thus (see [16] Theorem 2.3.12) standard. Since $D = E \cdot G \cdot E$, we can write $f_{\frac{M}{N}}$ as:

$$f_{\frac{M}{N}} = G^{n_1} \cdot E \cdot G^{n_2} \cdot E \cdots E \cdot G^{n_{2k+1}} \quad \text{where} \begin{cases} n_1, n_{2k+1} \geq 0 \\ n_2, \ldots, n_{2k} \geq 1 \\ k \geq 1. \end{cases}$$

Depending on the subindex n_i, there are three possibilities for the continued fraction expansion of $\phi_{\frac{M}{N}}$ (see [16] Theorem 2.3.25):

1. $\phi_{\frac{M}{N}} = [0; 1 + n_1, \overline{n_2, \ldots, n_{2k+1} + n_1}]$ if $n_{2k+1} > 0$.
2. $\phi_{\frac{M}{N}} = [0; 1, n_2, \overline{n_3, \ldots, n_{2k}, n_{2k+1} + n_1}]$ if $n_{2k+1} > 0$ and $n_1 = 0$.
3. $\phi_{\frac{M}{N}} = [0; 1 + n_1, \overline{n_2, \ldots, n_{2k}, n_1}]$ if $n_{2k+1} > 0$ and $n_1 > 0$.

Cases 2 and 3 are incompatible with the purely periodic, palindromic, continued expansion of $\phi_{\frac{M}{N}}$ (the second case would imply that whether $n_2 = 0$ or $n_k = 0$, and in the third case we would necessarily have that $n_1 = 1 + n_1$). Therefore the first case is the only possible. Hence we can compute the exponents n_i depending on the coefficients a_i of the continued fraction expansion:

$$\begin{aligned} &n_1 = a_1 - 1 \\ &n_i = a_i \quad \forall i = 2, \ldots, k \\ &n_{k+i} = a_{k+1-i} \quad \forall i = 1, \ldots, k \\ &n_{2k+1} = 1, \end{aligned}$$

and now we can compute $f_{\frac{M}{N}}$ with exponents depending on a_i $i = 1, \ldots, k$:

$$\begin{aligned} f_{\frac{M}{N}} &= G^{a_1-1} \cdot E \cdot G^{a_2} \cdots G^{a_k} \cdot E \cdot G^{a_k} \cdots G^{a_1} \cdot E \cdot G \\ &= G^{a_1-1} \cdot E \cdot G^{a_2} \cdots G^{a_k} \cdot D \cdot E \cdot G^{a_k-1} \cdots G^{a_1} \cdot D \cdot E = i_{\frac{M}{N}} \cdot j_{\frac{M}{N}}. \quad \square \end{aligned}$$

It is straightforward to prove the following corollary, where the standard morphism $f_{\frac{M*}{N}}$ fixing the characteristic word of slope $\phi^*_{\frac{M}{N}}$ is computed:

Corollary. *Let $\phi_{\frac{M}{N}} = \left[0; \overline{a_{\frac{M}{N}}, a^t_{\frac{M}{N}}}\right]$ be an odd mirror number and let $f_{\frac{M}{N}} = i_{\frac{M}{N}} \cdot j_{\frac{M}{N}}$ be the standard morphism which fixes it. In this situation we have that $f_{\frac{M*}{N}} = j_{\frac{M}{N}} \cdot i_{\frac{M}{N}}$ is the standard morphism that fixes the N-dual odd mirror number $\phi^*_{\frac{M}{N}}$. Moreover $i_{\frac{M}{N}}(c_{\phi^*_{\frac{M}{N}}}) = c_{\phi_{\frac{M}{N}}}$ and $j_{\frac{M}{N}}(c_{\phi_{\frac{M}{N}}}) = c_{\phi^*_{\frac{M}{N}}}$.*

Example (*Continuation*)**.** We compute the standard morphisms $f_{\frac{4}{7}}$ and $f_{\frac{2}{7}}$ which fix the characteristic words of slope $\phi_{\frac{4}{7}}$ and $\phi^*_{\frac{4}{7}}$. $i_{\frac{4}{7}} = G^0DG^3DE = DG^3DE$ and $j_{\frac{4}{7}} = G^2DGDE$. It follows that $f_{\frac{4}{7}} = DG^3DE \cdot G^2DGDE = DG^3D^3GDG$, and $f_{\frac{2}{7}} = G^2DGDE \cdot G^2DGDE = G^2DGDGD^3G$.

Now we can relate the characteristic words of odd mirror number slope with *harmonic words* as introduced in [9]. A central word u beginning with 0 (resp. beginning with 1) and having the periods p, q is harmonic if the following equality holds:

$$\frac{|u|_1 + 1}{|u|_0 + 1} = \frac{p}{q} \quad \left(\text{resp. if } \frac{|u|_0 + 1}{|u|_1 + 1} = \frac{p}{q} \text{ holds.}\right).$$

Equivalently, u is harmonic iff the ratio of the frequencies of the letters 0 and 1 in the Christoffel word $0u1$ coincides with the ratio of the periods of the central word u. Moreover, an infinite word $w = a_1a_2\ldots$ is said to be harmonic, if it has infinitely many harmonic prefixes.

Proposition 10. *The characteristic word $c_{\phi_{\frac{M}{N}/(1+\frac{M}{N})}} = G(c_{\phi_{\frac{M}{N}}})$ of cutting slope $\phi_{\frac{M}{N}}$ and mechanical slope $\phi_{\frac{M}{N}/(1+\frac{M}{N})}$ is a harmonic word.*

Proof. One just has to notice that the continued fraction expansion of $\phi_{\frac{M}{N}}$ has infinite many palindromic prefixes, which expand to sesqui-palindromic prefixes of the associated directive word. Proposition 3.5 in [9] completes the proof. $\square$

Example (*Continuation*)**.** The odd mirror number $\phi_{\frac{4}{7}} = \left[0; \overline{1, 1, 3, 3, 1, 1}\right] = \frac{-3+\sqrt{34}}{5}$ has period $(1, 1, 3, 3, 1, 1)$ with positive integers as coefficients. It can be associated with a binary path segment of length $1 + 1 + 3 + 3 + 1 + 1 = 10$ on the Stern–Brocot tree, and can be represented by the sesqui-palindrome 0100011101. This serves as the directive word for an harmonic word of length 128:

$$\begin{aligned} PAL(0100011101) = {} & 0100100100101001001001010010010010100100100101001001001010010 \\ & 01001010010010010100100100100101001001001010010010010100100100010. \end{aligned}$$

The period $(3, 1, 1, 1, 1, 3)$ of the dual odd mirror number $\phi_{\frac{2}{7}} = \left[0; \overline{3, 1, 1, 1, 1, 3}\right] = \frac{-5+\sqrt{34}}{3}$ yields the sesqui-palindrome 0001010111. This serves as the directive word for an harmonic word of length 104:

$$\begin{aligned} PAL(0001010111) = {} & 0001000010001000010000100010000100010000100001000100 \\ & 0010001000010000100010000100010000100001000100001000. \end{aligned}$$

Note, that each of the harmonic words $PAL(0100011101)$ and $PAL(0001010111)$ is a self-dual central word. But in connection of our notion of 7-duality we relate them with one another as prefixes of two infinite harmonic words. In accordance with Proposition 8, the infinite 10-periodic words $(0100011101)^\infty$ and $(0001010111)^\infty$ are the directive words of infinite harmonic words. Their cutting slopes are the odd mirror numbers $\phi_{\frac{4}{7}}$ and $\phi_{\frac{2}{7}}$ respectively.

4. Further investigations

4.1. Even mirror numbers

A natural question arises after we studied the set of odd mirror numbers: what can we say about the *even mirror numbers*? That is, given a Farey number $\frac{M}{N} = [0; a_1, \ldots, a_k + 1]$, we could define the even mirror number $\overline{\phi}_{\frac{M}{N}}$ related to $\frac{M}{N}$ as

$$\overline{\phi}_{\frac{M}{N}} = \begin{cases} [0; \overline{a_1, \ldots, a_k + 1, 1 + a_k, \ldots, a_1}] & \text{if } k \text{ is even} \\ [0; \overline{a_1, \ldots, a_k, 1, 1, a_k, \ldots, a_1}] & \text{if } k \text{ is odd.} \end{cases}$$

It is very easy to construct a complete binary tree such as its left half corresponds with the odd semi-periods whereas its right half corresponds with the even semi-periods. It is enough to take $\{1\}$ (the associated irrational number is the decimal part of the golden number) as the root of the tree. Thereafter we use the same recursive laws given in Remark 1. The resulting tree codifies the whole set of mirror numbers; see Fig. 6.

Let $\frac{M}{N} = [0; a_1, \ldots, a_k + 1] \in \mathfrak{F}$ and $\alpha = L^{a_1} R^{a_2} \cdots A^{a_k} = \left(\begin{smallmatrix} M_2 & M_1 \\ N_2 & N_1 \end{smallmatrix}\right)$ the associated matrix of $SL_2(\mathbb{N})$. We can consider the even mirror number as the solution of the equation

$$\mu[Id]\left(\overline{\phi}_{\frac{M}{N}}\right) = \overline{\phi}_{\frac{M}{N}},$$

and now we read the properties of the Möbius transform *backwards* to find the first matrix $B \in SL_2(\mathbb{N})$ such that $\mu[B](\overline{\phi}_{\frac{M}{N}}) = \overline{\phi}_{\frac{M}{N}}$. One can check, regardless of the parity of k, that:

$$\mu[\alpha R\,(\alpha R)^t]\left(\overline{\phi}_{\frac{M}{N}}\right) = \overline{\phi}_{\frac{M}{N}}.$$

Thus, the function that one should investigate the even mirror numbers is:

$$\begin{aligned} \overline{\Psi}\left(h, \frac{M}{N}\right) &= \mu\left[(\alpha R)^t\right](h) = \mu\left[\begin{pmatrix} M_1 & N_1 \\ M & N \end{pmatrix}\right](h) = \frac{M_1 h + N_1}{Mh + N} \\ &= 1 - \frac{M_2 h + N_2}{Mh + N} = 1 - \Psi\left(h, \frac{M}{N}\right). \end{aligned}$$

Many natural questions arise at this point: what properties of the set of odd mirror numbers also hold for the even mirror numbers? Which is the geometrical procedure that generates them? What is the relation between odd and even mirror numbers, and what kind of numerical and word theoretical relations does the whole set of mirror number fulfill?

4.2. Derivative word and the relation between $\phi_{\frac{M}{N}}$ and $\phi^*_{\frac{M}{N}}$

Given a Sturmian word w, a factor u of w such that both $0u$ and $1u$ are also factors of w is called left special factor of w. It is well known that it has $N + 1$ distinct factors of length N, for every $N > 0$. Hence, for every N, there is exactly one left special factor, u_N of w of length N.

Let u_N denote the left special factor of a Sturmian word $w = a_1 a_2 \ldots$, and let i_k be the place of the first letter of the kth occurrence of u_N in w. That is, let $a_{i_k} a_{i_k+1} \ldots a_{i_k+N-1} = u_N$ be the kth occurrence of the left special factor u_N in w.

Definition. We call N-derivative of the Sturmian word $w = a_1 a_2 \ldots$ to the word $\Delta_N(w)$ defined by:

$$\begin{aligned} \Delta_N(w) : \mathbb{N} &\longrightarrow \{0, 1\} \\ k &\longmapsto a_{i_k - 1}. \end{aligned}$$

The N-derivative of a Sturmian word w encodes the sequence of those letters which immediately precede the left special factor of length N in w.

The following conjecture extends the result of Proposition 8 and relates the N-derivative $\Delta_N\left(c_{\phi_{\frac{M}{N}}}\right)$ of the characteristic word $c_{\phi_{\frac{M}{N}}}$ of (odd mirror number) slope $\phi_{\frac{M}{N}}$ with the characteristic word of dual slope $\phi^*_{\frac{M}{N}}$:

Conjecture. *For any odd mirror number $\phi_{\frac{M}{N}}$ the N-derivative $E\left(\Delta_N\left(c_{\phi_{\frac{M}{N}}}\right)\right)$ of the characteristic word $c_{\phi_{\frac{M}{N}}}$ with exchanged letters coincides with the harmonic word of cutting slope $\phi^*_{\frac{M}{N}}$, i.e. for every $\frac{M}{N} \in \mathfrak{F}$ the following equality holds:*

$$E\left(\Delta_N\left(c_{\phi_{\frac{M}{N}}}\right)\right) = G\left(c_{\phi^*_{\frac{M}{N}}}\right).$$

Example (*Continuation*)**.** The characteristic word $c_{\phi_{\frac{4}{7}}}$ of mechanical slope $\phi_{\frac{4}{7}}$ is:

$$\begin{aligned} c_{\phi_{\frac{4}{7}}} = {} & 101010\mathbf{1}101010\mathbf{1}101010\mathbf{1}1\mathbf{0}101010\mathbf{1}101010\mathbf{1}101010\mathbf{1}101010\mathbf{1}1\mathbf{0}101010 \\ & \mathbf{1}101010\mathbf{1}101010\mathbf{1}1\mathbf{0}101010\mathbf{1}101010\mathbf{1}101010\mathbf{1}101010\mathbf{1}1\ldots \end{aligned}$$

The left special factor of length 7 is 1010101, and the letters which immediately precede it are highlighted in boldface. Thus, the 7-derivative of $c_{\phi_{\frac{4}{7}}}$ is:

$$\Delta_7\left(c_{\phi_{\frac{4}{7}}}\right) = 11101111011101111\ldots$$

Now, if we compute $G\left(c_{\phi^*_{\frac{4}{7}}}\right) = G\left(c_{\phi_{\frac{2}{7}}}\right)$ we obtain:

$$G\left(c_{\phi_{\frac{2}{7}}}\right) = 00010000100010000\ldots$$

which coincides with $E\left(\Delta_7\left(c_{\phi_{\frac{4}{7}}}\right)\right)$.

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